An Inter-temporal Model of Consumption and Saving

Macroeconomics: Economic Cycles, Frictions and Policy

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Outline

Preferences

- We now assume that consumers live for 2 periods
- Consumers' preferences are given by:

$$\mathcal{U} = u(c_1) + \beta u(c_2) \tag{1}$$

- $\beta \in (0, 1)$ is called the discount factor — measures consumers' impatience
- $u(\cdot)$ is strictly increasing and strictly concave $\rightarrow u' > 0$ and $u'' < 0$
- Consumers take as given their income in each period, $\{y_1, y_2\}$ (partial equilibrium)
- In every period consumers can use their income for consumption or saving
- The consumption good is not storable — in order to save/borrow, an individual needs to find a counter-party to take the other side of the transaction
- No siblings, no altruistic motives and no uncertainty about death $\rightarrow$ no saving in period 2

Preferences

- Define the marginal rate of substitution as:

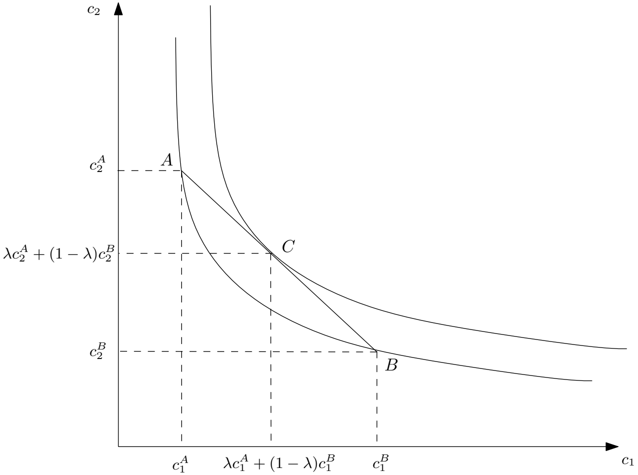
$$\mathsf{MRS}_{c_1,c_2} \equiv \frac{\mathcal{U}_{c_1}(c_1,c_2)}{\mathcal{U}_{c_2}(c_1,c_2)} = \frac{u'(c_1)}{\beta u'(c_2)} \tag{2}$$

- Mathematically, MRS_{c_1,c_2} is equal to minus the slope of the indifference curve:

$$\begin{aligned} \mathrm{d}\mathcal{U} = 0 \quad \Rightarrow \quad & \mathcal{U}_{c_1}(c_1,c_2)\mathrm{d}c_1 + \mathcal{U}_{c_2}(c_1,c_2)\mathrm{d}c_2 = 0 \\ & \frac{\mathrm{d}c_2}{\mathrm{d}c_1} = -\frac{\mathcal{U}_{c_1}(c_1,c_2)}{\mathcal{U}_{c_2}(c_1,c_2)} = \frac{u'(c_1)}{\beta u'(c_2)} \end{aligned} \tag{3}$$

- It measures a consumer's willingness to substitute consumption over time
- if an individual is consuming a lot in period 1 but very little in period 2 $\rightarrow u'(c_1)$ is very small and $\beta u'(c_2)$ is very large
- This implies that it would take a very large amount of c_1 to convince the individual to reduce c_2 — even by a bit

Desire for consumption smoothing



- Individuals prefer consumption bundle (c_1^C, c_2^C) to both bundles (c_1^A, c_2^A) and (c_1^B, c_2^B), where $c_t^C = \lambda c_t^A + (1-\lambda)c_t^B$, for $t = 1, 2$
- This means that individuals prefer to smooth consumption over time

Budget constraint

- There is a perfect capital market which consumers consumers can use to borrow/lend at a given interest rate, $r > 0$
- Budget constraints for each period are given by:

$$c_1 + s_1 = y_1 \qquad (t = 1)$$
$$c_2 = y_2 + (1 + r)s_1 \qquad (t = 2)$$

- Note that because the consumer can borrow, saving can potentially be negative
- Combining the two budget constraints into one by using the fact that $s_1 = y_1 - c_1$ and substituting it into period 2's budget constraint we obtain:

$$c_2 = y_2 + (1 + r)[y_1 - c_1] \qquad (4)$$

dividing by $(1 + r)$ on both sides

and moving c_1 and c_2 terms to the left hand side $\Rightarrow$

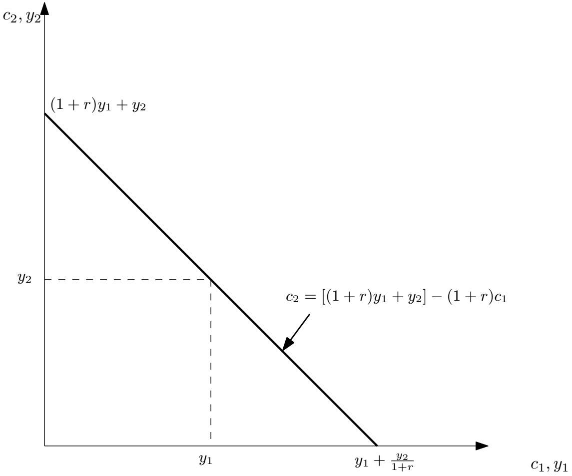
$$c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r} \qquad (5)$$

- The present discounted value of consumption cannot exceed the present discounted value of income
- Since $u' > 0 \rightarrow$ budget constraint is always binding $\Rightarrow$ the consumer spends all his life-time income

Budget constraint

- To plot the budget constraint in the $\{c_1, c_2\}$-space, rewrite it leaving only c_2 on the left hand side:

$$c_2 = [(1+r)y_1 + y_2] - (1+r)c_1 \tag{6}$$



Budget constraint

- Even though consumers can borrow in period 1 (and therefore set $s_1 < 0$), there's a limit on how much debt they can take
- We require them to be able to repay all their debt in period 2
- The maximum amount that a consumer can borrow in period 1 is an amount such that he would have to use all his income in period 2 to repay the debt (i.e. $c_2 = 0$)
- If $c_2 = 0 \Rightarrow$ (using the budget constraint)

$$c_1 = y_1 + \frac{y_2}{1+r} \tag{7}$$

which means that s_1 is given by:

$$s_1 = y_1 - c_1 \tag{8}$$

$$s_1 = -\frac{y_2}{1+r} \tag{9}$$

- The maximum amount that a consumer can borrow in period 1 is the present discounted value of their period 2's income, $y_2/(1+r)$
- The maximum amount that an individual can consume in period 2 is $y_2 + (1+r)y_1$, if he saves all his period 1's income, y_1

Consumer's problem

- The problem of the consumer is:

$$\max_{c_1, c_2} \quad \mathcal{U} = u(c_1) + \beta u(c_2) \tag{10}$$

s.t.:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{11}$$

- The Lagrangian is:

$$\mathcal{L} = u(c_1) + \beta u(c_2) - \lambda \left[c_1 + \frac{c_2}{1+r} - y_1 - \frac{y_2}{1+r} \right] \tag{12}$$

- Take the FOC of (12) with respect to c_1, c_2 and λ and set them equal to zero to find the optimal consumption in each period:

$$[c_1]: \quad u'(c_1) - \lambda = 0 \tag{13}$$

$$[c_2]: \quad \beta u'(c_2) - \frac{\lambda}{1+r} = 0 \tag{14}$$

$$[\lambda]: \quad -c_1 - \frac{c_2}{1+r} + y_1 + \frac{y_2}{1+r} = 0 \tag{15}$$

Consumer's problem

- Dividing equation (7) by (8) to eliminate λ and rearranging terms we obtain the Euler equation

$$\boxed{u'(c_1) = \beta(1+r)u'(c_2)} \tag{16}$$

- This is exactly the same as the optimality condition in the 1-period model
- The marginal rate of substitution between two goods (MRS) has to be equal to the ratio of the prices between the goods. The price of consumption in period 1 is 1 and the price of consumption in period 2 is $1/(1+r)$.

$$\text{MRS}_{c_1,c_2} = \frac{u'(c_1)}{\beta u'(c_2)} = \frac{p_1}{p_2} = \frac{1}{1/1+r} = 1+r \tag{17}$$

- The Euler equation is a necessary condition for the optimal consumption plan
- It means that if the consumption bundle is optimal, the individual cannot increase his utility by rearranging consumption between periods 1 and 2 (while still satisfying the budget constraint)

Implications of the Euler equation

- Euler equation: $u'(c_1) = \beta(1+r)u'(c_2)$
- Remember that since u is strictly concave $\rightarrow u'(c)$ is strictly decreasing
- The behavior of consumption over time depends on the rate of time preference relative to the real interest rate, r:

$$\beta(1+r) > 1 \quad \Rightarrow \quad u'(c_1) > u'(c_2) \quad \Rightarrow c_1 < c_2 \tag{18}$$

$$\beta(1+r) < 1 \quad \Rightarrow \quad u'(c_1) < u'(c_2) \quad \Rightarrow c_1 > c_2 \tag{19}$$

$$\beta(1+r) = 1 \quad \Rightarrow \quad u'(c_1) = u'(c_2) \quad \Rightarrow c_1 = c_2 \tag{20}$$

- If $\beta(1+r) = 1$ we observe perfect consumption smoothing

An example

- Assume that the consumer's utility in every period takes the form:

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma}, & \sigma > 0 \text{ and } \sigma \neq 1 \\ \ln(c), & \sigma = 1 \end{cases} \tag{21}$$

- Thus $u'(c) = c^{-\sigma}$
- Plugging this into the Euler equation, we have:

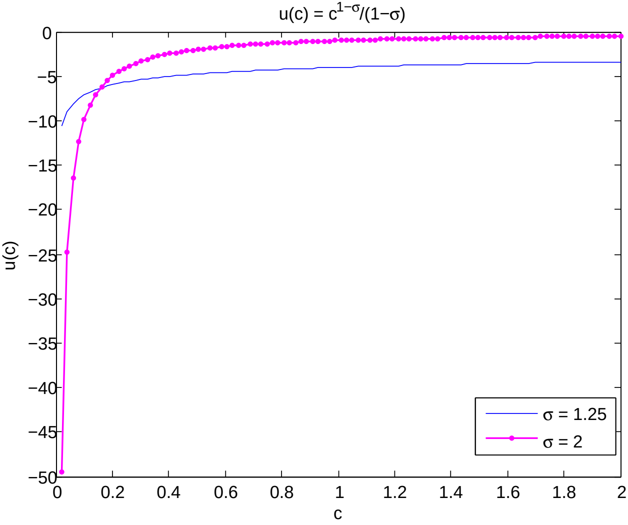
$$c_1^{-\sigma} = \beta(1+r)c_2^{-\sigma} \tag{22}$$

rearranging terms

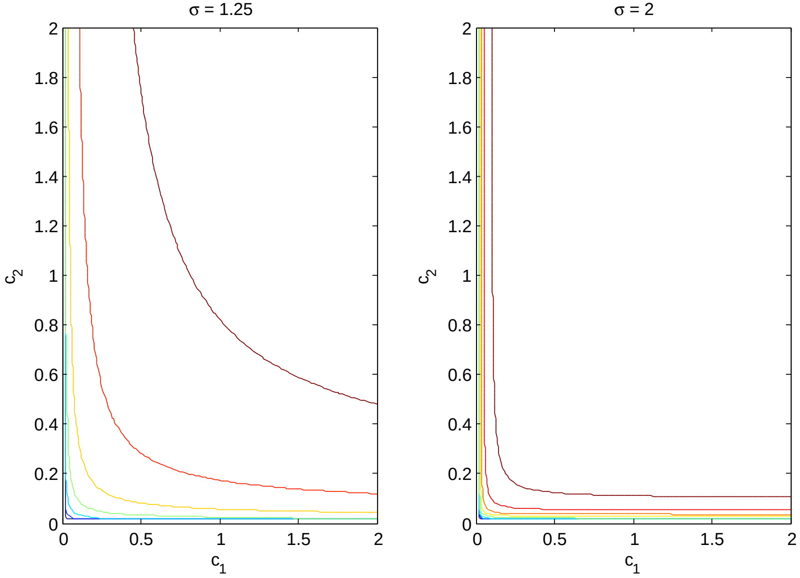
$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \beta(1+r) \tag{23}$$

$$\frac{c_2}{c_1} = [\beta(1+r)]^{\frac{1}{\sigma}} \tag{24}$$

Power utility function



Power utility function: indifference curves



An example

- Using equation (24) in the budget constraint (5), we find the optimal consumption for $t = 1, 2$ and savings

$$c_1^* = \frac{1}{1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1-\sigma}{\sigma}}} \left[y_1 + \frac{y_2}{1+r} \right] \tag{25}$$

$$c_2^* = \frac{[\beta(1+r)]^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1-\sigma}{\sigma}}} \left[y_1 + \frac{y_2}{1+r} \right] \tag{26}$$

$$s_1^* = y_1 - c_1^* = \frac{1}{1 + \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1-\sigma}{\sigma}}} \left[\beta^{\frac{1}{\sigma}} (1+r)^{\frac{1-\sigma}{\sigma}} y_1 - \frac{y_2}{1+r} \right] \tag{27}$$

The inter-temporal elasticity of substitution

- Define the inter-temporal elasticity of substitution, IES, as

$$\boxed{\text{IES} = \frac{\mathsf{d} \ln(c_2/c_1)}{\mathsf{d} \ln(1+r)}} \tag{28}$$

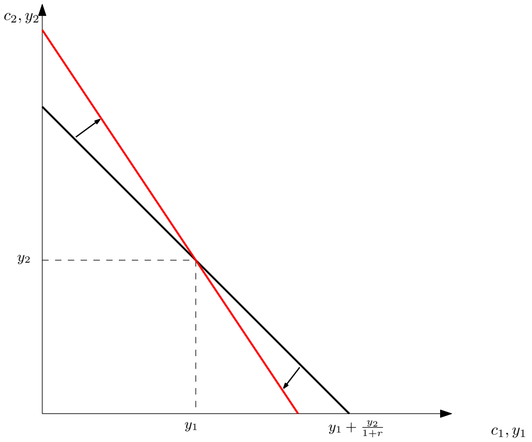
- Which you can easily calculate by taking logs of equation (24):

$$\begin{aligned} \ln(c_2/c_1) &= (1/\sigma)\Big[\ln(\beta) + \ln(1+r)\Big] \\ \Rightarrow \quad \text{IES} &= \frac{1}{\sigma} \end{aligned} \tag{29}$$

- If $1/\sigma$ is lower (i.e. if σ is large) $\rightarrow$ the utility function is highly curved $\rightarrow$ stronger consumption smoothing
- The consumer will be less prone to pursue inter-temporal substitution $\rightarrow$ less tilting of the consumption path
- In other words, the consumer requires large changes in interest rates in order to accept a less smoothed pattern of consumption

Increase in r

- An increase in r rotates the budget constraint clockwise along the income point (y_1, y_2) (see the red line in the figure below)



- As any price change, the increase in r has both a substitution and an income effect

Increase in r

- Substitution effect: an increase in r increases the price of consumption in period 1 relative to period 2. \$1 less of consumption in period 1 is producing a larger amount of consumption in period 2
- The substitution effect of an increase in r produces a reduction in c_1 (equivalently, an increase in s_1) — regardless of whether the consumer is a borrower or a lender
- Income effect: this will depend on whether the consumer was a borrower $(s_1 < 0)$ or a lender $(s_1 > 0)$ before r increased:
  - If the consumer was a lender, the increase in r increases the amount of income he has available in period 2 $\rightarrow$ increase consumption in both periods. The effect of an increase in r on savings is ambiguous
  - For the same reason, a consumer that was initially borrowing money to finance consumption in period 1 reduces his consumption in both periods. For a borrower, an increase in r univocally increases savings

More than 2 periods

- Assume that the consumer lives for an infinite number of periods
- An individual's recursive budget constraint is the same as before:

$$c_t + s_{t+1} = (1+r)s_t + y_t, \quad \forall t = 0, 1, \ldots \tag{30}$$

- Iterating forward on the budget constraint allows us to write the consumer's problem as follows:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{31}$$

s.t.:

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \tag{32}$$

- Verify (writing the Lagrangian of this problem and solving for the FOC), that the same Euler equation as in the 2-period model still holds

The Permanent Income Hypothesis (PIH)

- The classical Keynesian consumption function took the following form:

$$C_t = A + mpc \cdot Y_t, \quad mpc \in (0, 1) \tag{33}$$

- This functional form assumes that consumers are myopic
- What about retirement? Do consumers react the same to permanent and transitory shocks?

Definition (The Permanent Income Hypothesis)

Consumption is proportional to permanent income in each period. Permanent income represents the best estimate, given currently available information, of the individual's lifetime resources (both financial and human resources)

- The key insight is that consumers decide how much to consume keeping in mind their future prospects

PIH

- Let's examine some specific cases:
- Let $\beta = (1+r)^{-1}$. $\Rightarrow u'(c_t) = u'(c_{t+1}) \Rightarrow c_t = c_{t+1} = c$, $\forall t$
- Plugging $c_t = c$ for all t in the budget constraint, you get:

$$c = \left(\frac{r}{1+r}\right) \underbrace{\left[a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right]}_{\text{permanent income}} \tag{34}$$

- Consumption in each period is a constant fraction $r/(1+r)$ of discounted lifetime wealth or permanent income
- Consumption doesn't respond a lot to changes in current income:

$$\frac{\partial c_t}{\partial y_t} = \frac{r}{1+r} \tag{35}$$

- The income stream given by $\{y_t\}_{t=0}^{\infty}$ could be highly variable, but the consumer is able to smooth consumption perfectly by borrowing and lending in a perfect capital market
- Find the optimal consumption pattern $\{c_t\}$ when $u(c) = \ln(c)$ — no assumptions are required about $\beta(1+r)$

Recursive formulation

- We can represent the consumer problem in a recursive way using s_t as our state variable and s_{t+1} as our control:

$$v(s) = \max_{s' \leq (1+r)[w+s]} \left\{ u\left[s + y - \frac{s'}{1+r} \right] + \beta v(s') \right\}. \tag{36}$$

- Remember $s' = (1+r)\Big[s + y - c\Big]$
- From the FOC and the envelope condition we obtain the Euler equation:

$$\begin{aligned} [s'] : \quad \frac{u'(c)}{1+r} &= \beta v_s(s') \\ v_s(s') &= u'(c') \\ u'(c_t) &= \beta(1+r)u'(c_{t+1}) \end{aligned} \tag{Euler equation}$$

Introducing uncertainty

- What happens if $\{y_t\}$ is not known with certainty?
- The consumer problem looks very similar to what we saw before:

$$v(s,y) = \max_{s'} \left\{ u\left[s + y - \frac{s'}{1+r}\right] + \beta \mathbb{E}_{y'|y} v(s', y') \right\} \tag{37}$$

- We just need to make y an exogenous state variable in the consumer's problem
- Accordingly, the Euler equation now looks like this:

$$u'(c_t) = \beta(1+r)\mathbb{E}_t[u'(c_{t+1})] \tag{38}$$

- Following Hall (1978), assume that $u(c_t) = -1/2(\overline{c} - c_t)^2$
- $\Rightarrow u'(c_t) = \overline{c} - c_t$. Plugging $u'(c_t)$ and $u'(c_{t+1})$ in the Euler equation:

$$\overline{c} - c_t = \beta(1+r)\mathbb{E}_t[\overline{c} - c_{t+1}] \tag{39}$$

Introducing uncertainty

- Thus, we have:

$$\mathbb{E}_t[c_{t+1}] = \underbrace{\left[1 - \frac{1}{\beta(1+r)}\right]\bar{c}}_{\gamma_0} + \underbrace{\left[\frac{1}{\beta(1+r)}\right]}_{\gamma} c_t$$

$$\mathbb{E}_t[c_{t+1}] = \gamma_0 + \gamma c_t \tag{40}$$

- Let ε be a random variable with $\mathbb{E}[\varepsilon_t] = 0 \ \forall t$. Then

$$c_{t+1} = \gamma_0 + \gamma c_t + \varepsilon_{t+1} \tag{41}$$

- Consumption follows a martingale $\rightarrow$ its changes are unpredictable
- No information available in period t apart from the level of consumption c_t, helps predict future consumption, c_{t+1}. In particular, income or wealth in periods t or earlier are irrelevant once c_t is known

Introducing uncertainty

- For simplicity, let $\beta(1+r) = 1 \Rightarrow c_{t+1} = c_t + \varepsilon_{t+1}$, or $\Delta c_{t+1} = \varepsilon_{t+1}$
- Let's start from period 0 (you can start the recursion in any period t)

$$
\begin{aligned}
c_1 &= c_0 + \varepsilon_1, \\
c_2 = c_1 + \varepsilon_2 &= c_0 + \varepsilon_1 + \varepsilon_2, \\
\Rightarrow c_t &= c_0 + \sum_{\tau=1}^{T} \varepsilon_\tau, \\
\Rightarrow \mathbb{E}_0[c_t] &= c_0
\end{aligned}
$$

- Taking conditional expectations on the budget constraint yields:

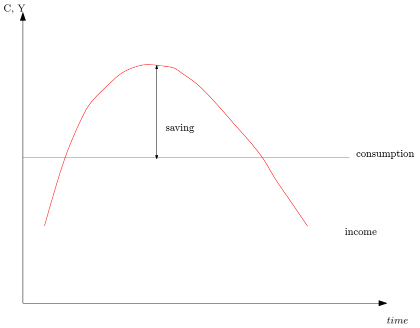
$$
\begin{aligned}
\mathbb{E}_0\left[\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}\right] &= \mathbb{E}_0\left[s_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t}\right], \\
c_0 &= \left(\frac{r}{1+r}\right) s_0 + \left(\frac{r}{1+r}\right)\left[\sum_{t=0}^{\infty} \frac{\mathbb{E}_0[y_t]}{(1+r)^t}\right] \qquad (42)
\end{aligned}
$$

- Only unexpected changes to permanent income should affect consumption

A little bit of historical context

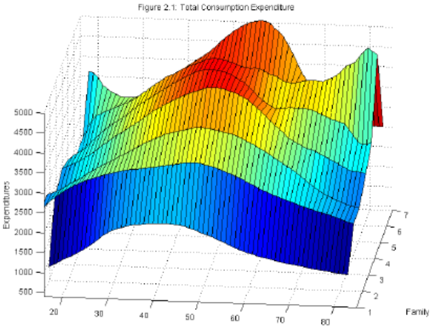
- Until the 1970s, PI was constructed as a weighted avg. of lagged measured income $\Rightarrow$ this is specification is completely ad-hoc
- ...but it fitted the data reasonably well
- Until the first oil shock of 1973!
- In 1976 Robert Lucas published one of the most influential papers in macroeconomics that completely doomed the old consumption function [Lucas, R. E. (1976) "Econometric policy evaluation: a critique" *Carnegie-Rochester Conference Series in Public Policy* 1: 19–46]
- Lucas argued that there is no reason to expect the ad-hoc formulations of PI (or any other economic variable) to be stable
- Since consumption depends on expected future income, individuals will incorporate any new information they can get into their forecasts of future income
- The Lucas critique is embedded in the dynamic programming approach to macro that we have taken in this module

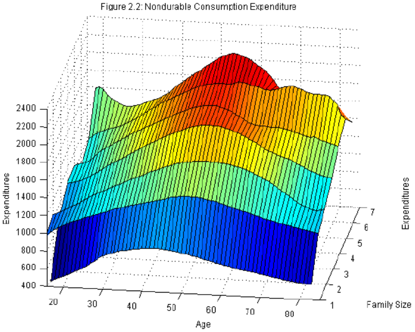
Disposable income over the life-cycle



Source: Attanasio (1999)

Consumption over the life-cycle





Source: Fernandez-Villaverde & Krueger (2001)

- Note that even after controlling for family size, consumption over the life-cycle looks hump-shaped
- It seems to closely track current income

Asymmetric information

- Assume that endowments are such that a fraction $1 - a \in (0, 1)$ of the population, receives 2nd-period endowment $y_2 = 0$; the rest of the population receives endowment $y_2 > 0$
- The key difference with our previous model is that only each individual knows his/her true type. Imagine that based on your preferences you would like to be a lender. Would you individually lend your period 1 endowment to a random person in the economy?
- Let's introduce a banking sector. It receives deposits from savers (for which it pays interest rate r_1) and lends funds to individuals (charging interest rate r_2)
- The advantage of the bank is that it can minimize the risk of default by diversifying. That is, based on the law of large numbers, on average a fraction $1 - a$ of the bank loans would be defaulted on
- Under perfect competition, the bank makes zero profits (π):

$$\pi = aL(1 + r_2) - L(1 + r_1) = L[a(1 + r_2) - (1 + r_1)] = 0 \tag{43}$$

- Which implies that: $r_2 = \frac{1+r_1}{a} - 1 \quad \Rightarrow r_2 > r_1$
- Each good borrower pays a default premium on a bank loan $r_2 - r_1$, and this difference increases as the share of good borrowers a in the economy $\downarrow$

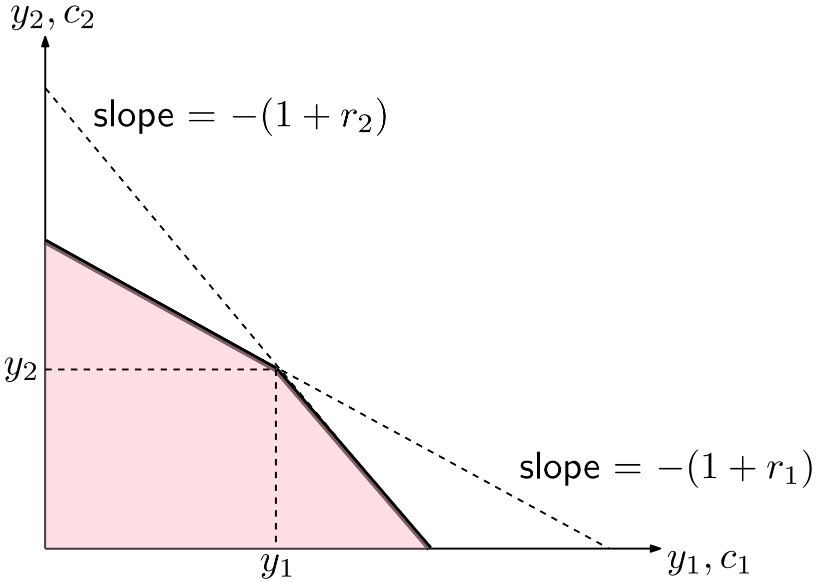
Budget constraint under asymmetric information

- The set of budget constraints for an individual in our 2-period model now becomes:

$$c_1 + s_1 = y_1 \qquad (t = 1)$$
$$c_2 = y_2 + s_1(1 + r_1) \qquad (s_1 \geq 0)$$
$$c_2 = y_2 + s_1(1 + r_2) \qquad (s_1 < 0)$$

- Notice that the budget set is given by the inner hull of the two budget constraints as shown in the figure in the next slide
- How does an increase in the perceived likelihood of default affect consumers in this economy?

Budget constraint under asymmetric information



Limited commitment

- Even if borrowers can repay their debts, they would be better off not doing so and 'running away' with the money, so to speak
- We have assumed so far that there is perfect enforcement of loan contracts. However, recovering a debt payment might be easier in countries with better institutions
- One way around this problem is to require borrowers to post collateral when they ask for a loan. A typical example of this is a mortgage loan
- Imagine that consumers in our economy own a house of size H, which can be sold in period 2 for a price p. For simplicity assume that the house cannot be sold in $t = 1$ (its an illiquid asset)
- The budget constraint for an individual is now given by:

$$c_1 + s_1 = y_1 \qquad (t = 1)$$
$$c_2 = y_2 + s_1(1 + r_1) + pH \qquad (t = 2)$$

- $\Rightarrow c_1 + \frac{c_2}{1+r} = y_1 = \frac{y_2+pH}{1+r}$

Limited commitment

- Lenders will be willing to lend an amount such that the borrower will find it beneficial to repay the loan (incentive-compatible loan)
- The amount borrowed by the consumer needs to satisfy:

$$-s_1(1+r) \leq pH \tag{44}$$

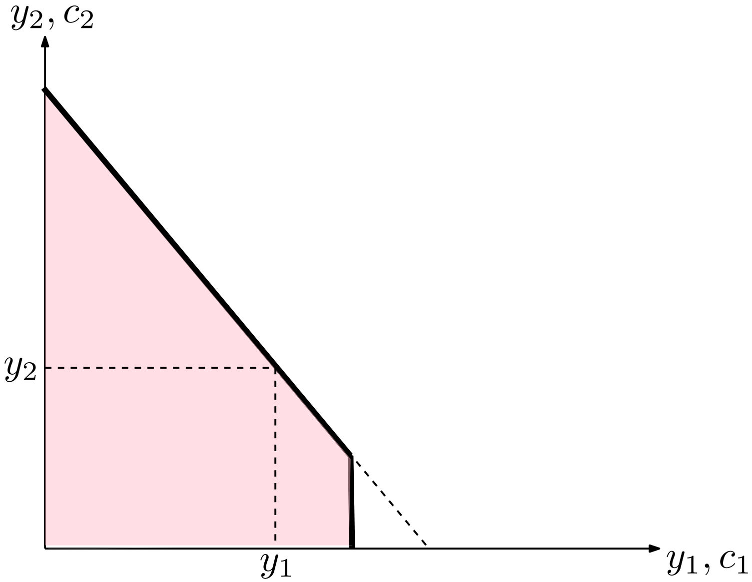
- Plugging this into the budget constraint for period 1:

$$c_1 = y_1 - s_1 \tag{45}$$

$$c_1 \leq y_1 + \frac{pH}{1+r} \tag{46}$$

- This again induces a kink in the individual's budget constraint
- How does a fall in house prices affect the budget constraint?

Budget constraint under limited commitment



Key questions:

- Write down explicitly the decision problem of the individual when he faces an exogenous income stream. Discuss the main parameters parameters of the model
- What is the economic intuition of having indifference curves that are convex to the origin in the 2-period model of consumption?
- Derive the lifetime budget constraint that the individual faces. What is its economic interpretation?
- Write down the Lagrangian characterizing the individual's problem in the 2-period consumption model. Derive the Euler equation and provide an economic interpretation for it
- What is the economic intuition of the inter-temporal elasticity of substitution (IES)?
- Describe what happens to optimal consumption and savings when y_1 or y_2 increases
- Describe what happens to optimal consumption and savings when r increases. Discuss the income and substitution effects of a change in real interest rates
- Describe what happens to the labor supply of the individual if the wage he receives in the market increases. Discuss the income and substitution effects of the wage change

Key questions

- What is the Permanent Income Hypothesis (PIH)?
- How does an increase in the share of the population perceived to be a ‘bad’ credit risk affect the interest rates that banks charge? How does this affect good-risk individuals?
- How does a fall in the value of collateral affect the budget constraint in the case of credit market failures due to limited commitment?
- Are credit market imperfections important to explain individual consumption patterns over an individual’s life-cycle?

References

- Williamson, S. *Macroeconomics*, Chapters 9 and 10
- Gross, D. B. and N. S. Souleles (2002) "Do Liquidity Constraints and Interest Rates Matter for Consumer Behavior? Evidence from Credit Card Data," *Quarterly Journal of Economics* 117: 149-185
- Hsieh, C.-T. (2003) "Do Consumers React to Anticipated Income Shocks? Evidence from the Alaska Permanent Fund," *American Economic Review* 93: 397-405